

Punjab University
 Journal of Mathematics (ISSN 1016-2526)
 Received on 26 July 2010
 Revised on 24 May 2011
 Accepted on 4 July 2011

Jordan Left Derivations on Lie Ideals of Prime Γ -rings

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Abstract. Let M be a 2-torsion free prime Γ -ring. Let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \rightarrow M$ is an additive mapping such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

AMS (MOS) Subject Classification Codes: 03E72, 54A40, 54B15

Key Words: n -torsion free, Lie ideals, Jordan left derivations, Prime Γ -rings.

1. INTRODUCTION

Let M and Γ be additive abelian groups. M is said to be a Γ -ring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (sending (x, α, y) into $x\alpha y$) such that

$$(a) \quad (x + y)\alpha z = x\alpha z + y\alpha z,$$

$$x(\alpha + \beta)y = x\alpha y + x\beta y,$$

$$x\alpha(y + z) = x\alpha y + x\alpha z,$$

$$(b) \quad (x\alpha y)\beta z = x\alpha(y\beta z),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

A subset A of a Γ -ring M is a left(right) ideal of M if A is an additive subgroup of M and $M\Gamma A = \{m\alpha a : m \in M, \alpha \in \Gamma, a \in A\}$, $A\Gamma M$ is contained in A . The centre of M is denoted by $Z(M)$ which is defined by $Z(M) = \{m \in M : a\alpha m = m\alpha a, a \in M, \alpha \in \Gamma\}$. M is commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$ and $\alpha \in \Gamma$. M is prime if $a\Gamma M\Gamma b = 0$ with $a, b \in M$, then $a = 0$ or $b = 0$. We denote the commutator $x\alpha y - y\alpha x$ by $[x, y]_\alpha$. An additive subgroup U of M is said to be a Lie ideal of M if $[u, x]_\alpha \in U$, for all $u \in U, x \in M$ and $\alpha \in \Gamma$. M is n -torsion free if $nx = 0$, for $x \in M$ implies $x = 0$, where n is an integer. An additive mapping $d : M \rightarrow M$ is a derivation if $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$, a left derivation if $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$, a Jordan derivation if

$d(a\alpha a) = a\alpha d(a) + d(a)\alpha a$ and a Jordan left derivation if $d(a\alpha a) = 2a\alpha d(a)$, for all $a, b \in M$ and $\alpha \in \Gamma$.

Y.Ceven [3] worked on Jordan left derivations on completely prime Γ -rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ -ring that makes the Γ -ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ -ring is a left derivation on it. In this paper, he gave an example of Jordan left derivation for Γ -rings.

Mustafa Ascı and Sahin Ceran [6] studied on a nonzero left derivation d on a prime Γ -ring M for which M is commutative with the conditions $d(U) \subseteq U$ and $d^2(U) \subseteq Z$, where U is an ideal of M and Z is the centre of M . They also proved the commutativity of M by the nonzero left derivation d_1 and right derivation d_2 on M with the conditions $d_2(U) \subseteq U$ and $d_1 d_2(U) \subseteq Z$.

In [7], M.Sapanci and A.Nakajima defined a derivation and a Jordan derivation on Γ -rings and investigated a Jordan derivation on a certain type of completely prime Γ -ring which is a derivation. They also gave examples of a derivation and a Jordan derivation of Γ -rings.

M. Bresar and J.Vukman[2] showed that the existence of a nonzero Jordan left derivation of R into X implies R is commutative, where R is a ring and X is 2-torsion free and 3-torsion free left R -module. In [8], Jun and Kim proved their results without the property 3-torsion free.

Qing Deng [4] worked on Jordan left derivations d of prime ring R of characteristic not 2 into a nonzero faithful and prime left R -module X . He proved the commutativity of R with Jordan left derivation d .

Mohammad Ashraf and Nadeem-Ur-Rehman[1] studied on Lie ideals and Jordan left derivations of prime rings. They proved that if d is an additive mapping on a 2-torsion free prime ring R satisfying $d(u^2) = 2ud(u)$, for all $u \in U$, where U is a Lie ideal of R such that $u^2 \in U$, for all $u \in U$, then $d(uv) = ud(v) + vd(u)$, for all $u \in U$.

In our paper, we reviewed the results of Mohammad Ashraf and Nadeem-Ur-Rehman[1] in gamma rings. We show that if d is an additive mapping on a 2-torsion free prime Γ -ring M such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, where U is a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. To complete the proof of main result in commutative sense, we take a help from the book ‘Topics in ring theory’ of Herstein[5]. Finally, we showed that every Jordan left derivation on U is a left derivation.

Throughout this paper, we shall use the mark (*) for $a\alpha b\beta c = a\beta b\alpha c$, for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

In order to prove our main result, we shall state and prove some lemmas as primary results.

2. PRIMARY RESULTS

Lemma 1. *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free σ -prime Γ -ring M . Then there exists an ideal I of M such that $[I, M]_\alpha \subseteq U$ but $[I, M]_\alpha \not\subseteq Z(M)$.*

Proof. Since M is 2-torsion free and $U \not\subseteq Z(M)$, it follows from the results in [6] that $[U, U]_\alpha \neq 0$ and $[I, M]_\alpha \subseteq U$, where $I = I\alpha[U, U]_\alpha\alpha M \neq 0$ is an ideal of M generated by $[U, U]_\alpha$. Now, $U \not\subseteq Z(M)$ implies $[I, M]_\alpha \not\subseteq Z(M)$; for, if $[I, M]_\alpha \subseteq Z(M)$ then

$[I, [I, M]_\alpha]_\alpha = 0$, which gives $I \subseteq Z(M)$ and, since $I \neq 0$ is an ideal of M , so $M = Z(M)$. \square

Lemma 2. *Let $U \not\subseteq Z(M)$ be a Lie ideal of a 2-torsion free prime Γ -ring M which satisfies the condition (*) and $a, b \in M$ such that $a\alpha U\beta b = 0$. Then $a = 0$ or $b = 0$.*

Proof. Since M is 2-torsion free prime Γ -ring, there exists an ideal I of M such that $[I, M]_\alpha \subseteq U$ but $[I, M]_\alpha \not\subseteq Z(M)$, by Lemma 1. Now, taking $u \in U$, $e \in I$ and $m \in M$, we have $[e\alpha\alpha u, m]_\alpha \in [I, M]_\alpha \subseteq U$, and so

$$\begin{aligned} 0 &= a\alpha[e\alpha\alpha u, m]_\beta\beta b \\ &= a\alpha[e\alpha\alpha, m]_\alpha\beta u\beta b + a\alpha e\beta a\alpha[u, m]_\alpha\beta b, \text{ by (*)} \\ &= a\alpha[e\alpha\alpha, m]_\alpha\beta u\beta b, \text{ since } a\alpha[u, m]_\alpha \in a\alpha U\beta b \\ &= a\alpha e\alpha\alpha m\beta u\beta b - a\alpha m\alpha e\alpha\alpha\beta u\beta b \\ &= a\alpha e\alpha\alpha m\beta u\beta b - a\alpha m\alpha e\beta a\alpha u\beta b, \text{ by (*)} \\ &= a\alpha e\alpha\alpha m\beta u\beta b, \text{ by assumption.} \end{aligned}$$

Thus $a\alpha I\alpha\alpha M\beta U\beta b = 0$.

If $a \neq 0$ then $U\beta b = 0$, by the primeness of M . Now, if $u \in U$ and $m \in M$ then $u\alpha m - m\alpha u \in U$ and hence $(u\alpha m - m\alpha u)\beta b = 0$ implies $u\alpha m\beta b = 0$, that is $u\alpha M\beta b = 0$. Since $U \neq 0$, we must have $b = 0$. In the similar fashion, it can be shown that if $b \neq 0$ then $a = 0$. \square

Lemma 3. *Let M be a 2-torsion free prime Γ -ring and let U be a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \rightarrow M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then*

- $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u)$. Let M satisfy (*), then
- $d(u\alpha v\beta u) = u\alpha v\beta d(u) + 3u\alpha v\beta d(v) - v\alpha u\beta d(u)$,
- $d(u\alpha v\beta w + w\alpha v\beta u) = (u\alpha w + w\alpha u)\beta d(v) + 3u\alpha v\beta d(w) + 3w\alpha v\beta d(u) - v\alpha u\beta d(w) - v\alpha w\beta d(u)$,
- $[u, v]_\alpha\alpha u\beta d(u) = u\alpha[u, v]_\alpha\beta d(u)$
- $[u, v]_\alpha\beta(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0$,

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Proof. Since $u\alpha v + v\alpha u = (u + v)\alpha(u + v) - u\alpha u - v\alpha v$, we have $u\alpha v + v\alpha u \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$. Then $d(u\alpha v + v\alpha u) = d((u + v)\alpha(u + v)) - d(u\alpha u) - d(v\alpha v)$ with our hypothesis yields the required result.

Replacing v by $u\beta v + v\beta u$ in (a), we have

$$\begin{aligned} d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) &= \\ 2u\alpha d(u\beta v + v\beta u) + 2(u\beta v + v\beta u)\alpha d(u). & \end{aligned} \quad (2.1)$$

Since $u\alpha v + v\alpha u \in U$, by (*) we get

$$\begin{aligned} d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) &= \\ 4u\alpha u\beta d(v) + 6u\alpha v\beta d(u) + 2v\alpha u\beta d(u). & \end{aligned} \quad (2.2)$$

On the other hand

$$\begin{aligned} d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) &= \\ d(u\alpha u\beta v + v\beta u\alpha u) + 2d(u\alpha v\beta u) &= \\ 2u\alpha u\beta d(v) + 4v\alpha u\beta d(u) + 2d(u\alpha v\beta u). \end{aligned} \quad (2.3)$$

Combining (2.2) and (2.3) and using the condition that M is 2-torsion free, we obtain (b).

Replacing $u + w$ for u in (b) and using (*), we get

$$\begin{aligned} d((u + w)\alpha v\beta(u + w)) &= \\ u\alpha u\beta d(v) + w\alpha w\beta d(v) + (u\alpha w + w\alpha u)\beta d(v) + \\ 3u\alpha v\beta d(u) + 3u\alpha v\beta d(w) + 3w\alpha v\beta d(u) + w\alpha v\beta d(w) - \\ v\alpha u\beta d(u) - v\alpha u\beta d(w) - v\alpha w\beta d(u) - v\alpha w\beta d(w). \end{aligned} \quad (2.4)$$

On the other hand with (*), we have

$$\begin{aligned} d((u + w)\alpha v\beta(u + w)) &= \\ d(u\alpha v\beta u) + d(w\alpha v\beta w) + d(u\alpha v\beta w + w\alpha v\beta u) &= \\ u\alpha u\beta d(v) + 3u\alpha v\beta d(u) - v\alpha u\beta d(u) + w\alpha w\beta d(v) \\ + 3w\alpha v\beta d(w) - v\alpha w\beta d(w) + d(u\alpha v\beta w + w\alpha v\beta u). \end{aligned} \quad (2.5)$$

Combining (2.4) and (2.5), we obtain (c).

Since $u\alpha v + v\alpha u$ and $u\alpha v - v\alpha u$ are in U , we see that $2u\alpha v \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$. By hypothesis, we have $d((u\alpha v)\beta(u\alpha v)) = 2u\alpha v\beta d(u\alpha v)$.

Replacing w by $2u\beta v$ in (c) with (*) and the condition that M is 2-torsion free, we get

$$\begin{aligned} d(u\alpha v\beta(u\beta v) + (u\beta v)\alpha v\beta u) &= \\ (u\alpha u\beta v + u\alpha v\beta u)\beta d(v) + 3u\alpha v\beta d(u\beta v) + \\ 3u\alpha v\beta v\beta d(u) - v\alpha u\beta d(u\beta v) - v\alpha u\beta v\beta d(u). \end{aligned} \quad (2.6)$$

On the other hand with (*), we have

$$\begin{aligned} d(u\alpha v\beta(u\beta v) + (u\beta v)\alpha v\beta u) &= \\ d((u\beta v)\alpha(u\beta v) + u\alpha v\beta v\beta u) &= \\ 2u\alpha v\beta d(u\beta v) + 2u\alpha u\beta v\beta d(v) + \\ 3u\alpha v\beta v\beta d(u) - v\alpha v\beta u\beta d(u). \end{aligned} \quad (2.7)$$

Combining (2.6) and (2.7), we have

$$\begin{aligned} [u, v]_\alpha \beta d(u\beta v) &= \\ u\alpha [u, v]_\beta \beta d(v) + v\alpha [u, v]_\beta \beta d(u). \end{aligned} \quad (2.8)$$

Replacing $u + v$ for v in (2.8), we have

$$\begin{aligned} 2[u, v]_\alpha \beta u\beta d(u) + [u, v]_\alpha \beta d(u\beta v) &= \\ 2u\alpha [u, v]_\beta \beta d(u) + u\alpha [u, v]_\beta \beta d(v) + v\alpha [u, v]_\beta \beta d(u). \end{aligned} \quad (2.9)$$

From (2.8) and (2.9), we get (d).

Linearizing (d) on u , we have

$$\begin{aligned} [u, v]_{\alpha}\beta u\beta d(u) + [u, v]_{\alpha}\beta v\beta d(v) + [u, v]_{\alpha}\beta u\beta d(v) + [u, v]_{\alpha}\beta v\beta d(u) = & \quad (2.10) \\ \alpha[u, v]_{\beta}\beta d(u) + u\alpha[u, v]_{\beta}\beta d(v) + v\alpha[u, v]_{\beta}\beta d(u) + v\alpha[u, v]_{\beta}\beta d(v), \end{aligned}$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Application of (d) and (8) gives $[u, v]_{\alpha}\beta u\beta d(v) + [u, v]_{\alpha}\beta v\beta d(u) = [u, v]_{\alpha}\beta d(u\beta v)$ and hence $[u, v]_{\alpha}\beta(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0$, for all $u, v \in U$ and $\alpha, \beta \in \Gamma$. \square

Lemma 4. *Let M be a 2-torsion free Γ -ring satisfying (*) and U a Lie ideal of M such that $u\alpha u \in U$, for all $u \in U$ and $\alpha \in \Gamma$. If $d : M \rightarrow M$ is an additive mapping satisfying $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then*

- (a) $[u, v]_{\alpha}\beta d([u, v]_{\alpha}) = 0$,
(b) $(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) = 0$,
for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Proof. By Lemma 3(a) and Lemma 3(e), we get

$$d(u\alpha v + v\alpha u) = 2(u\alpha d(v) + v\alpha d(u)) \quad (2.11)$$

and

$$[u, v]_{\alpha}\beta(d(u\alpha v) - u\alpha d(v) - v\alpha d(u)) = 0. \quad (2.12)$$

Combining (2.11) and (2.12), we have

$$[u, v]_{\alpha}\beta(d(v\alpha u) - u\alpha d(v) - v\alpha d(u)) = 0. \quad (2.13)$$

Using (2.12) - (2.13), we get $[u, v]_{\alpha}\beta d([u, v]_{\alpha}) = 0$, for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

For any $u, v \in U$ and $\alpha, \beta \in \Gamma$, we have $d([u, v]_{\alpha}\beta[u, v]_{\alpha}) = 2[u, v]_{\alpha}\beta d([u, v]_{\alpha})$. By (a), we have

$$d([u, v]_{\alpha}\beta[u, v]_{\alpha}) = 0, \quad (2.14)$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

We have $2u\alpha v \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$.

Replacing u by $2u\beta v$ in $u\alpha v + v\alpha u \in U$ and $u\alpha v - v\alpha u \in U$ and adding the results and then using (*), we find that $4v\alpha u\beta v \in U$, for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Replacing $4v\alpha u\beta v$ for v in Lemma 3(a) and since M is 2-torsion free, we have

$$d(u\alpha(v\alpha u\beta v) + (v\alpha u\beta v)\alpha u) = 2(u\alpha d(v\alpha u\beta v) + v\alpha u\beta v\alpha d(u)). \quad (2.15)$$

Using (2.15) in (2.14) and then (*), we have

$$\begin{aligned} 0 = & \\ d(u\alpha(v\alpha u\beta v) + (v\alpha u\beta v)\alpha u) - d(u\alpha(v\alpha v)\beta u) - d(v\alpha(u\alpha u)\beta v) = & \\ 2(u\alpha d(v\alpha u\beta v) + v\alpha u\beta v\alpha d(u)) - u\alpha u\beta d(v\alpha v) & \\ - 3u\alpha v\alpha v\beta d(u) + v\alpha v\alpha u\beta d(u) - v\alpha v\beta d(u\alpha u) & \\ - 3v\alpha u\alpha u\beta d(v) + u\alpha u\alpha v\beta d(v) = & \\ - 3(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) & \\ - (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\beta d(u) & \end{aligned}$$

and hence

$$\begin{aligned} & 3(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) + \\ & (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\beta d(u) = 0, \end{aligned} \quad (2.16)$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Replacing u by $u + v$ in Lemma 3(d), we get

$$\begin{aligned} & (u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) - \\ & (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\beta d(u) = 0, \end{aligned} \quad (2.17)$$

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Combining (2.16) and (2.17), we obtain

$$(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\beta d(v) = 0. \quad (2.18)$$

By (2.17) and (2.18), we arrive at (b). \square

3. MAIN RESULT

The main result of this paper states as follows.

Theorem 5. *Let M be a 2-torsion free prime Γ -ring satisfying (*) and U a Lie ideal of M such that $u\alpha u \in U$, for all $u, v \in U$ and $\alpha \in \Gamma$. If $d : M \rightarrow M$ is an additive mapping such that $d(u\alpha u) = 2u\alpha d(u)$, for all $u \in U$ and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof. Suppose U is a commutative Lie ideal of M . Let $a \in U$ and $x \in M$. Then $[a, x]_\alpha \in U$ and so commutes with a . Now, for $x, y \in M$, we have $a\beta[a, x\alpha y]_\alpha = [a, x\alpha y]_\alpha\beta a$, for all $\alpha, \beta \in \Gamma$. Expanding $[a, x\alpha y]_\alpha$ as $[a, x]_\alpha\alpha y + x\alpha[a, y]_\alpha$ and using that a commutes with this, with $[a, x]_\alpha$ and with $[a, y]_\alpha$, we have $2[a, x]_\alpha\alpha[a, y]_\alpha = 0$ and so $[a, x]_\alpha\alpha[a, y]_\alpha = 0$, since M is 2-torsion free. Replacing y by $a\beta x$ in $[a, x]_\alpha\alpha[a, y]_\alpha = 0$ and then using (*), we have $[a, x]_\alpha\alpha M\beta[a, x]_\alpha = 0$, for all $x \in M$ and $\alpha, \beta \in \Gamma$. Since M is prime, $[a, x]_\alpha = 0$ and so $U \subset Z(M)$. Hence by Lemma 3(a), we have $2d(u\alpha v) = 2(u\alpha d(v) + v\alpha d(u))$. Since M is 2-torsion free, $d(u\alpha v) = u\alpha d(v) + v\alpha d(u)$.

We assume that U is a noncommutative Lie ideal of M .

Now, replacing u by $[u_1, w]_\alpha$ in Lemma 3(d), we get

$$\begin{aligned} & ([u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha v - 2[u_1, w]_\alpha\alpha v\alpha[u_1, w]_\alpha \\ & + v\alpha[u_1, w]_\alpha\alpha[u_1, w]_\alpha)\beta d([u_1, w]_\alpha) = 0, \end{aligned} \quad (3.1)$$

for all $u, v, u_1, w \in U$ and $\alpha, \beta \in \Gamma$.

Using Lemma 4(a) in (3.1), we get $[u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha v\beta d([u_1, w]_\alpha) = 0$

and so $[u_1, w]_\alpha\alpha[u_1, w]_\alpha\alpha U\beta d([u_1, w]_\alpha) = 0$.

Hence by Lemma 2, either $[u_1, w]_\alpha\alpha[u_1, w]_\alpha = 0$ or $d([u_1, w]_\alpha) = 0$.

If $d([u_1, w]_\alpha) = 0$ i.e. $d(u_1\alpha w) = d(w\alpha u_1)$, for all $u_1, w \in U$ and $\alpha \in \Gamma$, then by

Lemma 3(a) and the fact that M is 2-torsion free, we get $d(u_1\alpha w) = u_1\alpha d(w) + w\alpha d(u_1)$.

On the other hand let $[u_1, w]_\alpha\alpha[u_1, w]_\alpha = 0$, for some $u_1, w \in U$ and $\alpha \in \Gamma$.

Replacing v by $[u_1, w]_\alpha$ in Lemma 4(b), we get

$$\begin{aligned} & (u\alpha u\alpha[u_1, w]_\alpha)\beta d([u_1, w]_\alpha) \\ & - 2(u\alpha[u_1, w]_\alpha\alpha u)\beta d([u_1, w]_\alpha) + ([u_1, w]_\alpha\alpha u\alpha u)\beta d([u_1, w]_\alpha) = 0. \end{aligned} \quad (3.2)$$

Applying Lemma 4(a) in (3.2), we have

$$([u_1, w]_\alpha \alpha u \alpha u) \beta d([u_1, w]_\alpha) - 2(u \alpha [u_1, w]_\alpha \alpha u) \beta d([u_1, w]_\alpha) = 0, \quad (3.3)$$

for all $u \in U$ and $\alpha, \beta \in \Gamma$.

Linearizing (3.3) on u and using Lemma 4(b), we have

$$([u_1, w]_\alpha \alpha u \alpha v) \beta d([u_1, w]_\alpha) + ([u_1, w]_\alpha \alpha v \alpha u) \beta d([u_1, w]_\alpha) - 2((u \alpha [u_1, w]_\alpha \alpha v) + (v \alpha [u_1, w]_\alpha \alpha u)) \beta d([u_1, w]_\alpha) = 0, \quad (3.4)$$

for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Replacing u by $2u\beta v_1$ in (3.4) and then using the fact the M is 2-torsion free and (*), we have

$$[u_1, w]_\alpha \alpha u \beta v_1 \alpha v \beta d([u_1, w]_\alpha) + [u_1, w]_\alpha \alpha v \beta u \alpha v_1 \beta d([u_1, w]_\alpha) - 2(u \alpha v_1 \beta [u_1, w]_\alpha \alpha v + v \alpha [u_1, w]_\alpha \alpha u \beta v_1) \beta d([u_1, w]_\alpha) = 0. \quad (3.5)$$

Further, replacing v_1 by $[u_1, w]_\alpha$ in (3.5) and then using Lemma 4(b), $[u_1, w]_\alpha \alpha [u_1, w]_\alpha = 0$ and (*),

we get $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha \alpha v \beta d([u_1, w]_\alpha) = 0$

i.e., $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha \alpha U \beta d([u_1, w]_\alpha) = 0$, for all $u \in U$ and

$\alpha, \beta \in \Gamma$. By Lemma 2, either $d([u_1, w]_\alpha) = 0$ or $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha = 0$.

If $d([u_1, w]_\alpha) = 0$, then by the same argument as above we get the required result. On the other hand, if $[u_1, w]_\alpha \alpha u \beta [u_1, w]_\alpha = 0$, for all $u \in U$ and $\alpha, \beta \in \Gamma$, then by Lemma 2, we have $[u_1, w]_\alpha = 0$. Further, by Lemma 3(a) and the fact that M is 2-torsion free, we have $d(u_1 \alpha w) = u_1 \alpha d(w) + w \alpha d(u_1)$. Hence in both cases, we find that $d(u \alpha v) = u \alpha d(v) + v \alpha d(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. The proof is thus complete. \square

Corollary 6. *Let M be a 2-torsion free prime Γ -rings and $d : M \rightarrow M$ a Jordan left derivation. Then d is a left derivation on M .*

Proof. If M is commutative, then $u \alpha v = v \alpha u$, for all $u, v \in M$ and $\alpha \in \Gamma$, and so by Lemma 3(a), we have $d(u \alpha v) = u \alpha d(v) + v \alpha d(u)$, for all $u, v \in M$ and $\alpha \in \Gamma$. If M is noncommutative, then by Theorem 5, we have $d(u \alpha v) = u \alpha d(v) + v \alpha d(u)$, for all $u, v \in M$ and $\alpha \in \Gamma$. \square

Acknowledgement. The authors are grateful to the referees for their valuable instructions.

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